

Spherically symmetric problem on the brane and galactic rotation curves

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We investigate the braneworld model with induced gravity to clarify the role of the cross-over length scale ℓ in the possible explanation of the dark-matter phenomenon in astrophysics and in cosmology. Observations of the 21 cm line from neutral hydrogen clouds in spiral galaxies reveal that the rotational velocities remain nearly constant at a value $v_c \sim 10^{-3} - 10^{-4}$ in the units of the speed of light in the region of the galactic halo. Using the smallness of v_c , we develop a perturbative scheme for reconstructing the metric in a galactic halo. In the leading order of expansion in v_c , at the distances $r \gtrsim v_c \ell$, our result reproduces that obtained in the Randall–Sundrum braneworld model. This inequality is satisfied in a real spiral galaxy such as our Milky Way for distances $r \sim 3$ kpc, at which the rotational velocity curve becomes flat, $v_c \sim 7 \times 10^{-4}$, if $\ell \lesssim 2$ Mpc. The gravitational situation in this case can be approximately described by the Einstein equations with the so-called Weyl fluid playing the role of dark matter. In the region near the gravitating body, we derive a closed system of equations for static spherically symmetric situation under the approximation of zero anisotropic stress of the Weyl fluid. We find the Schwarzschild metric to be an approximate vacuum solution of these equations at distances $r \lesssim \sqrt[3]{r_g \ell^2}$. The value $\ell \lesssim 2$ Mpc complies well with the solar-system tests. At the same time, in cosmology, a low-density braneworld with ℓ of this order of magnitude can mimic the expansion properties of the high-density LCDM (Λ + cold dark matter) universe at late times. Combined observations of galactic rotation curves and gravitational lensing can possibly discriminate between the higher-dimensional effects and dark matter.

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I. INTRODUCTION

The idea that our familiar four-dimensional space-time is a hypersurface (brane) of a five-dimensional space-time (bulk) [1, 2, 3] was under detailed elaboration during the last decade. According to this braneworld scenario, all matter and gauge interactions reside on the brane, while gravity can propagate in the whole five-dimensional space-time. An observer in this scenario is in direct contact only with the induced metric on the brane.

A ponderable argument in favor of the braneworld theory is the ability to solve outstanding problems of modern cosmology and astrophysics. The generally accepted LCDM (Λ + cold dark matter) model is in a good agreement with most of experimental data; however, up to now, no non-gravitational evidence for dark matter has been reliably found. It was argued [4, 5, 6, 7] (see also [8] for a somewhat different approach) that a modified theory of gravity based on the Randall–Sundrum braneworld scenario [2] can explain the observations of the galactic rotation curves, while observations of the gravitational lensing can possibly discriminate between the higher-dimensional effects and dark matter. In a recent work [9], the authors argued that the virial-theorem mass discrepancy in clusters of galaxies can also be accounted for in the Randall–Sundrum model. However, this version of braneworld model cannot explain the dark-matter phenomenon on the cosmological scale without introducing other components in the theory. At the same time, as shown in [10], a low-density braneworld *with induced gravity* can mimic the expansion properties of the LCDM model. In particular, a universe consisting solely of baryons with $\Omega_b \simeq 0.04$ can mimic the LCDM cosmology with a much larger ‘effective’ value of the matter density $\Omega_m \simeq 0.2\text{--}0.3$. This effect becomes possible owing to the presence of a fundamental cross-over length scale ℓ in braneworld models with induced gravity, which is absent in the Randall–Sundrum braneworld model.

The aim of this paper is to investigate the properties of the braneworld model with induced gravity to clarify further the role of the scale ℓ in the possible explanation of the dark-matter phenomenon in astrophysics as well as in cosmology. From the general consideration [10, 11], one expects the braneworld model to resemble the general relativity at the distances $r \lesssim \sqrt[3]{r_g \ell^2}$, thus giving an approximate Schwarzschild metric in vacuum with r_g in a role of the Schwarzschild radius. At large distances, the modified gravitational field equations can

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be used to explain the properties of the galactic rotation curves.

Observations of the 21 cm line from neutral hydrogen clouds in spiral galaxies reveal that the rotational velocities remain nearly constant in the halo region at values $v_c \sim 10^{-3} - 10^{-4}$ in the units of the speed of light. Starting from this observational fact, the exact metric in a galactic halo was reconstructed in frames of the Randall–Sundrum braneworld model in [6, 7], which was used to account for this phenomenon by the effects of higher-dimensional gravity without dark matter. We extend the results obtained in these works to the case of braneworld model with induced gravity. Using the smallness of v_c , we develop a perturbative scheme for reconstructing metric in a galactic halo and demonstrate that, in the leading order of expansion, at the distances $r \gtrsim v_c \ell$, our result reproduces that obtained for the Randall–Sundrum braneworld model in [6, 7]. The gravitational situation in this case can be approximately described by the Einstein equations with the traceless projection of the bulk Weyl tensor to the brane playing the role of an effective stress–energy tensor of galactic fluid. To give a prescription for verifying the scenario based on the braneworld models, we apply the results of [12], in which it was proposed to use the existing data on rotation curves and lensing measurements to constrain the equation of state of the galactic fluid.

To solve the braneworld equations at small distances to the gravitating source, one should impose some additional conditions. As first noted in [13], the nonlocality and nonclosure of the braneworld equations are connected with the projection C_{ab} of the bulk Weyl tensor to the brane. It, therefore, seems reasonable to impose certain restrictions on this tensor in order to obtain a closed system of equations on the brane. In this way, several classes of vacuum static spherically symmetric solutions on the brane were obtained for the Randall–Sundrum braneworld model [14, 15]. Solutions obtained by imposing direct restrictions on the form of the metric (such as $g_{tt} = -g_{rr}^{-1}$ [16]) also correspond to some evident conditions on C_{ab} . Finding the correct boundary condition of this kind, one can turn the braneworld model into a complete nonlinear *local* theory of gravity viable in all physical circumstances.¹

The simplest way of restricting the tensor C_{ab} consists in setting to zero its (appropriately defined) anisotropic stress.² This condition is fully compatible with all equations of

¹ Another possibilities to restrict the space of solutions of braneworld equations in a spherically symmetric case was discussed in [15, 17].

² Earlier attempts to simplify the braneworld equations by entirely neglecting the contribution from C_{ab} proved to be incorrect since this condition is incompatible with the equation that follows from the Bianchi

the theory and leads to a brane universe described by a modified theory of gravity with an additional invisible component (Weyl fluid) having nontrivial dynamics. A one-parameter family of boundary conditions (with zero anisotropic stress as a particular case) was successfully employed in [18] to derive an exact system of equations describing scalar cosmological perturbations on a generic braneworld with induced gravity.

In the present paper, we derive a closed system of equations for static³ spherically symmetric situation in the case of zero anisotropic stress in the tensor C_{ab} (the *minimal boundary condition* in the terminology of [18]). We find the Schwarzschild metric to be an approximate vacuum solution of these equations at distances $r \lesssim \sqrt[3]{r_g \ell^2}$, which justifies using of the minimal boundary condition in the vicinity of gravitating objects.

The paper is organized as follows. In the next section, we remind the field equations of the braneworld model with induced gravity. In Sec. III, we discuss the generic properties of the braneworld equations. In Sec. IV, we describe the cosmic mimicry model. In Sec. V, we reconstruct the metric in a galactic halo with flat rotation curves. In Sec. VI, we derive and discuss the system of equations for the static spherically symmetric case with the minimal conditions for the Weyl fluid. In Sec. VII, we find vacuum solutions in the weak field limit. In the last section, we present our conclusion.

II. EFFECTIVE FIELD EQUATIONS

We start from the simplest generic braneworld model described by the action

$$S = M^3 \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda) - 2 \int_{\text{brane}} K \right] + \int_{\text{brane}} (m^2 R - 2\sigma) + \int_{\text{brane}} L(h_{ab}, \phi). \quad (1)$$

Here, \mathcal{R} is the scalar curvature of the metric g_{ab} in the five-dimensional bulk, and R is the scalar curvature of the induced metric $h_{ab} = g_{ab} - n_a n_b$ on the brane (n^a is the vector field of the inner unit normal to the brane) which is assumed to be a boundary of the bulk space, and the notation and conventions of [20] are used. The quantity $K = h^{ab} K_{ab}$ is the trace of

identity [see Eq. (8) below]. It was first noted in [13] in the context of the Randall–Sundrum model that neglecting C_{ab} in the spherically symmetric situation leads to a severe restriction on the energy-momentum tensor of perfect fluid, namely, it only admits solutions with constant matter density (incompressible fluid). Below, we demonstrate that this result is valid also in our more general braneworld model with induced gravity.

³ For a discussion of the gravitational collapse on the brane see [19].

the symmetric tensor of extrinsic curvature K_{ab} of the brane. The symbol $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric h_{ab} . All integrations over the bulk and brane are taken with the corresponding natural volume elements. The symbols M and m denote the five-dimensional and four-dimensional Planck masses, respectively, Λ is the bulk cosmological constant, and σ is the brane tension.

Action (1) leads to the Einstein equation with cosmological constant in the bulk:

$$\mathcal{G}_{ab} + \Lambda g_{ab} = 0, \quad (2)$$

while the field equation on the brane is

$$m^2 G_{ab} + \sigma h_{ab} = T_{ab} + M^3 (K_{ab} - h_{ab} K), \quad (3)$$

where T_{ab} is the stress-energy tensor on the brane stemming from the last term in (1). The Einstein equation in the bulk (2) and the Gauss-Codazzi identities imply the conservation of the stress-energy tensor of matter on the brane, namely, $\nabla^a T_{ab} = 0$ (see, for example, [21]). Here, ∇_a denotes the covariant derivative on the brane associated with the induced metric h_{ab} .

By using the Gauss-Codazzi identities and projecting the field equations to the brane, one can obtain the following effective equation [10]:

$$G_{ab} + \frac{\Lambda_{\text{RS}}}{\beta + 1} h_{ab} = \left(\frac{\beta}{\beta + 1} \right) \frac{1}{m^2} T_{ab} + \frac{1}{\beta + 1} \left(\frac{1}{M^6} Q_{ab} - C_{ab} \right), \quad (4)$$

where

$$\beta = k\ell, \quad k = \frac{\sigma}{3M^3}, \quad \ell = \frac{2m^2}{M^3} \quad (5)$$

are convenient parameters of the braneworld model,

$$\Lambda_{\text{RS}} = \frac{\Lambda}{2} + \frac{\sigma^2}{3M^6} \quad (6)$$

is the value of the effective cosmological constant in the Randall-Sundrum model,

$$Q_{ab} = \frac{1}{3} E E_{ab} - E_{ac} E^c_b + \frac{1}{2} \left(E_{cd} E^{cd} - \frac{1}{3} E^2 \right) h_{ab} \quad (7)$$

is a quadratic expression with respect to the ‘bare’ Einstein equation $E_{ab} \equiv m^2 G_{ab} - T_{ab}$ on the brane, and $E = h^{ab} E_{ab}$. The symmetric traceless tensor $C_{ab} \equiv n^c n^d C_{acbd}$ is a projection

of the bulk Weyl tensor C_{abcd} which carries information about the gravitational field outside the brane. The tensor C_{ab} is not freely specifiable on the brane, but is related to the tensor Q_{ab} through the conservation equation

$$\nabla^a (Q_{ab} - M^6 C_{ab}) = 0, \quad (8)$$

which is a consequence of the Bianchi identity applied to (4).

The following famous braneworld theories are related to important subclasses of action (1):

1. The Randall–Sundrum model [2] is obtained after setting $m = 0$ in (1). The corresponding effective equation on the brane was derived in [13]:

$$G_{ab} + \Lambda_{\text{RS}} h_{ab} = \frac{2\sigma}{3M^6} T_{ab} + \frac{1}{M^6} S_{ab} - C_{ab}, \quad (9)$$

where

$$S_{ab} = \frac{1}{3} T T_{ab} - T_{ac} T^c_b + \frac{1}{2} \left(T_{cd} T^{cd} - \frac{1}{3} T^2 \right) h_{ab}. \quad (10)$$

In this model, only the right-hand side of the Einstein equation is modified.

2. The Dvali–Gabadadze–Porrati (DGP) model [3] corresponds to the special case where both the cosmological constant in the bulk and the brane tension vanish, i.e., $\Lambda = 0$ and $\sigma = 0$ in (1). The corresponding effective equation

$$G_{ab} = \frac{1}{M^6} Q_{ab} - C_{ab} \quad (11)$$

does not contain a linear contribution from the stress–energy tensor T_{ab} , which makes a perturbative analysis in this theory problematic.

3. Finally, general relativity, leading to the Λ CDM cosmological model, is formally obtained after setting $M = 0$ in (1). In this case, the effective equation $Q_{ab} = 0$ has an obvious solution

$$G_{ab} = \frac{1}{m^2} T_{ab}. \quad (12)$$

III. GENERIC PROPERTIES OF BRANEWORLD GRAVITY

The theory described by action (1) has two important scales, defined in (5), namely, the cross-over length scale ℓ , which describes the interplay between the bulk and brane gravity,

and the mass (energy) scale k , which determines the role of the brane tension in the dynamics of the brane. Thus, we can expect the properties of braneworld gravity to be different at different scales.

Indeed, looking at (4), we notice that expression (7) for Q_{ab} is quadratic in the curvature as well as in the stress–energy tensor. Therefore, in the region of high matter density and curvature, the tensor Q_{ab} dominates in (4), and the gravitational law is approximated by the ‘bare’ Einstein equations $m^2 G_{ab} - T_{ab} = 0$. And, vice versa, the contribution from Q_{ab} in (4) is insignificant on sufficiently large length scales, where the curvature is small, and the braneworld theory on those scales should again be approximated by the Einstein gravity with different gravitational constant. The tensor C_{ab} in this case plays the role of some effective stress–energy tensor.

To give some qualitative numerical estimates, we repeat here the reasoning of [10]. Consider the trace of the effective equation (4):

$$-R + \frac{4\Lambda_{\text{RS}}}{\beta + 1} - \left(\frac{\beta}{\beta + 1} \right) \frac{1}{m^2} T = \frac{1}{\beta + 1} \frac{1}{M^6} Q, \quad (13)$$

where the left-hand side contains terms which are linear in the curvature and in the stress–energy tensor while the right-hand side contains the quadratic term $Q = h^{ab} Q_{ab}$. This equation is closed on the brane in the sense that it does not contain the contribution from the traceless tensor C_{ab} .

Suppose that we are interested in the behaviour of gravity in the neighborhood of a spherically symmetric object with density ρ_s , total mass \mathcal{M}_s , and radius r_s . For simplicity, we assume that one can neglect the tensor C_{ab} and the effective cosmological constant in the neighborhood of the source. As regards the effective cosmological constant, this assumption is natural. Concerning the neglect of the tensor C_{ab} , we demonstrate the validity of this approximation in Sec. VII by considering the vacuum spherically symmetric situation in the braneworld model with C_{ab} restricted by a simple condition of vanishing anisotropic stress.

Within the source itself, we have two qualitatively different options: an approximate solution can be sought either neglecting the quadratic part or linear part of (4) and (13). We should choose the option that gives smaller error of approximation in (13). In the first case, neglecting the quadratic part and the effective cosmological constant, we have

$$G_{ab} - \left(\frac{\beta}{\beta + 1} \right) \frac{1}{m^2} T_{ab} \approx 0 \quad \Rightarrow \quad \frac{Q}{(\beta + 1)M^6} \sim \frac{\rho_s^2}{(\beta + 1)^3 M^6}. \quad (14)$$

In the second case, we neglect the linear part, so that

$$Q_{ab} \approx 0 \quad \Rightarrow \quad E_{ab} \approx 0 \quad \Rightarrow \quad R + \left(\frac{\beta}{\beta + 1} \right) \frac{1}{m^2} T \sim \frac{\rho_s}{(\beta + 1)m^2}. \quad (15)$$

The final expression on the right-hand side of (15) is smaller than the corresponding expression in (14) if

$$\rho_s > (\beta + 1)^2 \frac{M^6}{m^2} \quad \Rightarrow \quad r_s^3 < r_*^3 \sim \frac{\mathcal{M}_s \ell^2}{(\beta + 1)^2 m^2}, \quad (16)$$

where we used the relation $\mathcal{M}_s \sim \rho_s r_s^3$. Thus, we can expect that, in the neighborhood of the source, on distances smaller than r_* given by (16), the solution is determined mainly by the quadratic part Q_{ab} in (4), which means that it respects the ‘bare’ Einstein equation $m^2 G_{ab} - T_{ab} = 0$ to a high precision. This effect is sometimes described as the ‘gravity filter’ of the DGP model [3], which screens the scalar graviton in the neighborhood of the source making the gravity effectively Einsteinian. Some aspects of this interesting phenomenon were discussed in [22].

Expression (16) generalizes the length scale [11] of the DGP model, below which nonlinear effects become important, to the case of nonzero brane tension (nonzero β) and bulk cosmological constant satisfying the Randall–Sundrum constraint $\Lambda_{\text{RS}} = 0$. In order to comply with observations in the solar system, the value of ℓ should be chosen sufficiently large. In the next section, we will see that modeling the homogeneous dark matter phenomenon by cosmic mimicry requires $\beta \approx -5/4$, and in Sec. V we will show that matching the galactic rotation curves can be realized if $\ell \lesssim 2$ Mpc. For these parameter values, the value of r_* for the Sun is of the order of 1 pc, which is quite large so that the solar-system experiments are in accord with the general-relativistic expectations.

IV. COSMIC MIMICRY ON THE BRANE

In our paper [10], we described a braneworld model in which cosmological evolution proceeds similarly to that of the Friedmannian cosmology but with different values of the effective matter parameter Ω_m , or, equivalently, with different values of the effective gravitational constant G_N , at different cosmological epochs.

Specifically, for a spatially flat universe with zero background dark radiation ($C = 0$),

the cosmological equation stemming from (1) can be written as follows:

$$H^2 = \frac{\Lambda}{6} + \left[\sqrt{\frac{\rho - \rho_0}{3m^2} + \left(\sqrt{H_0^2 - \frac{\Lambda}{6}} \mp \frac{1}{\ell} \right)^2} \pm \frac{1}{\ell} \right]^2, \quad (17)$$

where ρ_0 and H_0 are the energy density of matter and Hubble parameter, respectively, at the present moment of time. The two signs in (17) correspond to two complementary possibilities for embedding the brane in the higher-dimensional (Schwarzschild-AdS) bulk.

In the effective mimicry scenario [10], the parameters $H_0^2 - \Lambda/6$ and $1/\ell^2$ are assumed to be of the same order, and much larger than the present matter-density term $\rho_0/3m^2$. The mimicry model has two regimes: one in the deep past (where the matter density was high) and another during the late-time evolution (where the matter density is low). In the deep past, we have

$$\frac{\rho - \rho_0}{3m^2} \gg \left(\sqrt{H_0^2 - \frac{\Lambda}{6}} \mp \frac{1}{\ell} \right)^2, \quad (18)$$

and the universe expands in a Friedmannian way

$$H^2 \approx \frac{\rho}{3m^2}. \quad (19)$$

During the late-time evolution, we have

$$\frac{\rho - \rho_0}{3m^2} \ll \left(\sqrt{H_0^2 - \frac{\Lambda}{6}} \mp \frac{1}{\ell} \right)^2, \quad (20)$$

and the expansion law is approximated by

$$H^2 \approx H_0^2 + \frac{\alpha}{\alpha \mp 1} \frac{\rho - \rho_0}{3m^2}, \quad (21)$$

where

$$\alpha = \ell \sqrt{H_0^2 - \frac{\Lambda}{6}} \quad (22)$$

is the parameter introduced in [10]. In the case of the cosmological branch with upper sign, it is assumed that the coefficient $\alpha/(\alpha - 1)$ in (21) and in similar expressions is always positive, i.e., α is assumed to be greater than unity in this case.

One can interpret the result (21) either as a renormalization of the effective gravitational constant relative to its value in the deep past or as a renormalization of the effective density parameter:

$$H^2 \approx H_0^2 + \frac{\rho^{\text{LCDM}} - \rho_0^{\text{LCDM}}}{3m^2}, \quad \rho^{\text{LCDM}} = \frac{\alpha}{\alpha \mp 1} \rho. \quad (23)$$

Remarkably, the behaviour of the Hubble parameter on the brane practically *coincides* with that in LCDM at low densities. This property was called ‘*cosmic mimicry*’ in [10]. A consequence of this is the fact that a low/high density braneworld consisting entirely of baryonic matter with $\Omega_b \simeq 0.04$ could easily masquerade as LCDM with a moderate value $\Omega_m^{\text{LCDM}} \simeq 0.2\text{--}0.3$. Thus, with the upper sign in (23), we can explain the phenomenon of homogeneous dark matter if suppose $\alpha \approx 5/4$. In this case, taking into account that the value of ℓ lies well below the Hubble scale, from (22) we obtain

$$|\Lambda|\ell^2 \approx 6\alpha^2 \approx 9. \quad (24)$$

The value of Λ should be negative for the expression under the square root in (22) to be positive. As was shown in [10], β is also negative in this case, and $|\beta| \approx \alpha$. The upper estimate $\ell \lesssim 2$ Mpc is obtained in the next section by considering galactic rotation curves. The range of redshifts over which this cosmic mimicry occurs is given by $0 \leq z \ll z_m$, where z_m in this case can be estimated to be $z_m \gtrsim 170$.

V. RECONSTRUCTION OF THE METRIC IN A GALACTIC HALO

In the previous section, we described how the phenomenon of homogeneous dark matter can be explained by cosmic mimicry in the braneworlds models with induced gravity. The question then arises whether the phenomenon of dark matter can also be accounted for in the inhomogeneous situations, in particular, on the scales of galaxies. This is the subject of the present section.

The problem of reconstructing the static metric and dark-matter halos in general relativity from the behavior of galactic rotation curves was investigated recently in [12, 23, 24]. A similar procedure was applied to the Randall–Sundrum braneworld model in [5, 6, 7], with a conclusion that it can explain the observations of the galactic rotation curves without dark matter. The Randall–Sundrum model, however, does not allow for the mimicry property and thus cannot explain the *homogeneous* dark-matter phenomenon. Hence, it is necessary to address this issue in frames of the induced-gravity model.

The traceless tensor C_{ab} drops out completely from the trace of (4), which is the only closed equation on the brane in the absence of any additional constraints on the tensor C_{ab}

[25]. In the vacuum, the trace of (4) reads

$$(1 + \beta)R + \frac{1}{4}\ell^2 \left(R_{ab}R^{ab} - \frac{1}{3}R^2 \right) - 4\Lambda_{\text{RS}} = 0. \quad (25)$$

One can use this equation to reconstruct the metric in the neighborhood of a galaxy by assuming approximate spherical symmetry and taking into account the qualitative behavior of rotation curves. In the Randall–Sundrum model [which corresponds to setting $\ell = 0$ and $\beta = 0$ in (25)], this procedure was done in [5, 6, 7].

The spherically symmetric metric, by which we approximate the situation in a galactic halo, has the form

$$ds^2 = -f(r)dt^2 + g^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (26)$$

In the region of the flat rotation curve, we have the condition

$$\phi'(r) = \frac{v_c^2}{r}, \quad (27)$$

where v_c is the rotation velocity, and $\phi(r) = \frac{1}{2} \log f(r)$. It can be integrated to give the g_{00} component of the metric:

$$f(r) = \left(\frac{r}{r_c} \right)^{2v_c^2}, \quad (28)$$

where r_c is the constant of integration. Substituting metric (26) with the function $\phi(r)$ given by (28) into equation (25), we obtain the following first-order nonlinear differential equation for the function $g(r)$:

$$\begin{aligned} & (rg' - 2g)^2 + 4rg' \left(1 - \frac{12(1 + \beta)r^2}{\ell^2} \right) - 8g \left(1 + \frac{6(1 + \beta)r^2}{\ell^2} \right) \\ & + 4 \left[1 + \frac{12r^2(1 + \beta - 2\Lambda_{\text{RS}}r^2)}{\ell^2} \right] \\ & + v_c^2 \left[-2r^2(g')^2 - 24rgg' + 8g^2 + 8rg' \left(1 - \frac{3(1 + \beta)r^2}{\ell^2} \right) - 8g \left(1 + \frac{6(1 + \beta)r^2}{\ell^2} \right) \right] \\ & + v_c^4 \left[(g')^2 - 12rgg' + 16g \left(1 - \frac{3(1 + \beta)r^2}{\ell^2} \right) \right] + 4v_c^6g(rg' - 4g) + 4v_c^8g^2 = 0. \end{aligned} \quad (29)$$

This equation looks rather complicated. However, we can take into account that the quantity $v_c^2 \sim 10^{-7}$ is very small for typical galaxies and thus represents a convenient parameter for asymptotic expansion. Therefore, we can look for the solution of (29) in the form of expansion in powers of v_c^2 :

$$g(r) = g_0(r) + v_c^2g_1(r) + v_c^4g_2(r) + \dots, \quad (30)$$

where the functions g_0, g_1, g_2, \dots , should be determined step by step by solving (29) with accuracy corresponding to the order of v_c at every step. First, we should find $g_0(r)$, which is the solution of (29) with all powers of v_c neglected. Introducing $\chi(r) = r g_0(r)$, we can rewrite the equation for $g_0(r)$ in the following form:

$$\ell^2 \left(\chi' - \frac{3\chi}{r} + 2 \right)^2 + 48r^2 [(1 + \beta)(1 - \chi') - 2\Lambda_{\text{RS}}r^2] = 0. \quad (31)$$

In the Randall–Sundrum limit, which implies $\ell = 0$ and $\beta = 0$, we should have $\chi_1(r) = r + a - 2\Lambda_{\text{RS}}r^3/3$ as a solution of (31), where a is some arbitrary constant of integration. On the other hand, the function $\chi_2(r) = r + br^3$, with another arbitrary constant b , sets expression in the first parentheses to zero. One can easily observe that the function $\chi(r) = r - 2\Lambda_{\text{RS}}r^3/3(1 + \beta)$ is an exact particular solution of (31), setting to zero both components on the left-hand side separately. Thus, for the zero-order approximation, we have

$$g_0(r) = 1 - \frac{2\Lambda_{\text{RS}}r^2}{3(1 + \beta)}. \quad (32)$$

The expression $2\Lambda_{\text{RS}}/(1 + \beta)$ here represents an analog of the cosmological constant in general relativity. At the scales at which galactic rotation curves are observed, the contribution from the cosmological constant should be negligibly small, namely

$$\frac{\Lambda_{\text{RS}}r^2}{|1 + \beta|} \ll 1, \quad (33)$$

thus allowing us to neglect it in the future. We should keep it for a moment, to estimate corrections in the second order of v_c .

Using (30) and (32), we find the differential equation for $g_1(r)$:

$$r g_1' + g_1 = -1 + \frac{\Lambda_{\text{RS}}\ell^2}{3(1 + \beta)^2} \left[1 - \frac{4\Lambda_{\text{RS}}r^2}{3(1 + \beta)} \left(1 - \frac{3(1 + \beta)^2}{\Lambda_{\text{RS}}\ell^2} \right) \right], \quad (34)$$

which can easily be integrated to give

$$g_1(r) = - \left(1 + \frac{C_1}{r} \right) + \frac{\Lambda_{\text{RS}}\ell^2}{3(1 + \beta)^2} \left[1 - \frac{4\Lambda_{\text{RS}}r^2}{9(1 + \beta)} \left(1 - \frac{3(1 + \beta)^2}{\Lambda_{\text{RS}}\ell^2} \right) \right], \quad (35)$$

where C_1 is some constant of integration having the dimension of length.

In the cosmic-mimicry theory which, as described in the previous section, explains the dark-matter phenomenon on the cosmological scale, the value of ℓ lies well below the cosmological length scale. Thus, we have

$$\frac{\Lambda_{\text{RS}}\ell^2}{(1 + \beta)^2} \ll 1. \quad (36)$$

Conditions (33) and (36) allow us to write the function $g(r)$ at this stage in the form

$$g(r) \approx 1 - \frac{2\Lambda_{\text{RS}} r^2}{3(1+\beta)} - v_c^2 \left(1 + \frac{C_1}{r}\right). \quad (37)$$

It is important to note that, under conditions (33) and (36), the difference between $g(r)$ in our model and the same quantity in the Randall–Sundrum braneworld model in this approximation in v_c^2 is only in the appearance of the factor $(1+\beta)$, which simply renormalizes the cosmological constant.

To find corrections of the forth order in v_c , we shall completely neglect the contribution from the term containing Λ_{RS} . This can be done under the condition

$$r^2 \ll \frac{v_c^2 |1+\beta|}{\Lambda_{\text{RS}}}, \quad (38)$$

which, obviously, is more restrictive than (33), but still is satisfied for a galaxy. The result is

$$g(r) \approx 1 - v_c^2 - \frac{v_c^2 C_1}{r} \left[1 - \frac{v_c^2}{2} \ln \frac{r}{|C_1|}\right] - \frac{\ell^2 v_c^4}{4(1+\beta)r^2} \left(1 - \frac{C_1}{2r} + \frac{C_1^2}{4r^2}\right), \quad (39)$$

in which the constant of integration C_1 is renormalized as compared with the previous approximation.

Although the last term in (39) is proportional to v_c^4 , at too small distances it can turn out to be larger than v_c^2 . This will obviously destroy our perturbative analysis. To avoid such situation, we restrict ourselves to the region⁴ $r^2 \gg \ell^2 v_c^2 / |1+\beta|$.

Leaving only corrections of the leading order in (39), we can write

$$g(r) \approx 1 - v_c^2 \left(1 + \frac{C_1}{r}\right), \quad (40)$$

which is a general reconstruction of the metric in our braneworld model in the region

$$\frac{\ell^2 v_c^2}{|1+\beta|} \ll r^2 \ll \frac{v_c^2 |1+\beta|}{\Lambda_{\text{RS}}} \quad (41)$$

with condition (28).

In order to satisfy the left inequality in (41) in a real spiral galaxy such as our Milky Way with the distance $r \sim 3$ kpc at which the rotation velocity becomes constant of magnitude

⁴ For simplicity, we assume the length scale $|C_1|$ to satisfy the condition $|C_1| \lesssim v_c \ell / \sqrt{|1+\beta|}$, so that $|C_1|/r$ is small. However, we would like to note here an interesting possibility that a large constant $|C_1|$ determines another interesting scale in a galactic halo.

$v_c \sim 7 \times 10^{-4}$, we should have $\ell \lesssim \sqrt{|1+\beta|} v_c^{-1} r \simeq 2$ Mpc, where we took into account the mimicry estimate $\beta = -\alpha \approx -5/4$ (see the end of Sec. IV). In this paper, we restrict our investigation to this case. For magnitudes of the cross-over length scale ℓ essentially larger than this value, one cannot use the perturbation expansion in powers of v_c^2 , and it is necessary to solve the nonlinear equation (29).

We can substitute (28), (40) into (4) to compute the tensor C_{ab} . The result is

$$C^t_t \approx \frac{(1+\beta)v_c^2}{r^2}, \quad C^r_r \approx \frac{(1+\beta)v_c^2}{r^2} \left(\frac{C_1}{r} - 1 \right), \quad C^\theta_\theta = C^\varphi_\varphi \approx -\frac{(1+\beta)v_c^2 C_1}{2r^3}, \quad (42)$$

where we have neglected the contribution from terms of order v_c^4 , which is correct in region (41). Again, we observe that, in the leading order of expansion, our result reproduces that obtained for the Randall–Sundrum braneworld model in [6, 7]. This is what we have expected from general consideration of Sec. III: on sufficiently large length scales, the contribution from the quadratic expression Q_{ab} in (4) is negligibly small. The gravitational situation in this case can be approximately described by the Einstein equations with the tensor C_{ab} playing the role of some effective stress–energy tensor \tilde{T}_{ab} :

$$8\pi G_N \tilde{T}_{ab} \equiv -\frac{1}{1+\beta} C_{ab} = G_{ab}, \quad (43)$$

where G_N is the gravitational constant adopted by an observer on these scales.

Using (42) and (43), we determine the effective energy density $\tilde{\rho}(r)$ and radial and transverse pressures $\tilde{p}_r(r)$ and $\tilde{p}_t(r)$, respectively, inside the galactic halo:

$$8\pi G_N \tilde{\rho}(r) \approx \frac{v_c^2}{r^2}, \quad 8\pi G_N \tilde{p}_r(r) \approx \frac{v_c^2}{r^2} \left(1 - \frac{C_1}{r} \right), \quad 8\pi G_N \tilde{p}_t(r) \approx \frac{v_c^2}{r^2} \frac{C_1}{2r}. \quad (44)$$

We note that $\tilde{p}_r(r) \neq \tilde{p}_t(r)$, which indicates the presence of a significant anisotropic stress in the tensor C_{ab} [see definition (47) below] on galactic scales.

It was argued in [12] that combined observations of galactic rotation curves and gravitational lensing can yield the profile of pressure inside a galactic halo. The authors of [12] proposed to use the pseudo-masses $m_{\text{RC}}(r)$ and $m_{\text{lens}}(r)$, which can be obtained for the same galactic halo from the treatment of rotation curves and gravitational lensing in the context of dark-matter paradigm (which implies negligible pressure), to define the χ -factor which quantifies the deviation from the predictions of the CDM model:

$$\chi[\omega(r)] = \frac{m'_{\text{lens}}(r)}{m'_{\text{RC}}(r)} = \frac{2 + 3\omega(r)}{2 + 6\omega(r)}, \quad (45)$$

where

$$\omega(r) = \frac{\tilde{p}_r(r) + 2\tilde{p}_t(r)}{3\tilde{\rho}(r)}. \quad (46)$$

For the braneworld model, using (44), one obtains $\omega \approx 1/3$ and $\chi \approx 3/4$, meaning that deflection angles, which can be computed in our model, will be around 75% of the usual predictions based on dark matter. This result coincides with the analysis made in [5].

VI. MINIMAL CONDITION FOR THE WEYL FLUID ON SMALL SCALES

In the previous section, we have shown that the contribution of Q_{ab} to (4) is negligibly small on sufficiently large spatial scales, which makes it possible to explain the flatness of galactic rotation curves as an effect of higher-dimensional gravity. On such scales, the braneworld theory reduces to the Einstein gravity with the tensor C_{ab} playing the role of some effective stress–energy tensor.

To demonstrate that the contribution from C_{ab} to the gravitational dynamics is insignificant at small spatial scales, we should solve Eq. (4) at small distances from the source of gravity. The problem arising here is that equation (4) is not closed on the brane in the sense that the dynamics of the symmetric traceless tensor C_{ab} on the brane is not determined by the dynamics of matter alone. An approximation usually taken by several authors to overcome this problem consists in imposing certain conditions on the tensor C_{ab} directly on the brane so as to close Eq. (4). Such conditions on the tensor C_{ab} should be compatible with the conservation equation (8) and should also leave the braneworld theory compatible with Einstein’s general relativity in a wide range of physical situations.

In general, the tensor C_{ab} can be decomposed through an arbitrary normalized timelike vector field u^a on the brane (see [26]):

$$m^2 C_{ab} = \frac{\rho_{\mathcal{C}}}{3} (h_{ab} + 4u_a u_b) + 2v_{(a}^{\mathcal{C}} u_{b)} + \pi_{ab}^{\mathcal{C}}. \quad (47)$$

Here, the covector $v_a^{\mathcal{C}}$ and traceless symmetric tensor $\pi_{ab}^{\mathcal{C}}$ are both orthogonal to u^a . The tensor C_{ab} in this case is regarded as the stress–energy tensor of an ideal fluid with equation of state like that for radiation but with nontrivial dynamics described by Eq. (8). In the literature, it is called ‘Weyl fluid’ [27] and, in the cosmological context, ‘dark radiation’ [28]. The quantity $\rho_{\mathcal{C}}$ has the meaning of its density, $v_a^{\mathcal{C}}$ is its momentum transfer, and $\pi_{ab}^{\mathcal{C}}$ is its anisotropic stress, as measured by observers following the world lines tangent to u^a . The

stress–energy of this ideal fluid is not conserved due to the presence of the source term Q_{ab} in (8).

Equation (8) gives evolution equations for the components $\rho_{\mathcal{C}}$ and $v_{\mathcal{C}}^a$. However, there are no evolution equations for the tensor fields $\pi_{ab}^{\mathcal{C}}$ on the brane, which is a manifestation of the nonlocality of the physical situation from the braneworld viewpoint. Thus, boundary conditions for the brane–bulk system can be specified by imposing additional conditions on the tensor C_{ab} .

As the first condition on the tensor field C_{ab} , we demand that it has a normalized timelike eigenvector field $u_{\mathcal{C}}^a$, so that $C^a_b u_{\mathcal{C}}^b \propto u_{\mathcal{C}}^a$. Then this vector field can be used as vector u^a in decomposition (47), which was not specified up to now:

$$m^2 C_{ab} = \frac{\rho_{\mathcal{C}}}{3} (h_{ab} + 4u_{\mathcal{C}a}^{\mathcal{C}} u_{\mathcal{C}b}^{\mathcal{C}}) + \pi_{ab}^{\mathcal{C}}. \quad (48)$$

Note that the covector component $v_a^{\mathcal{C}}$ vanishes in this case because of the eigenvector property of $u_{\mathcal{C}}^a$. The Weyl fluid is now described by the timelike eigenvector field $u_{\mathcal{C}}^a$, which can be interpreted as its four-velocity, and by the quantity $\rho_{\mathcal{C}}$, which has a meaning of its rest-frame density.

Now, in solving the spherically symmetric problem in the neighborhood of the source, we assume that the anisotropic stress can be neglected in the natural decomposition (48):

$$\pi_{ab}^{\mathcal{C}} = 0. \quad (49)$$

This is a *minimal boundary condition* for the brane–bulk system in the terminology of [18]. Under condition (49), the normalized eigenvector field $u_{\mathcal{C}}^a$ is unique modulo orientation.

Equation (8) gives the evolution equations for the components $\rho_{\mathcal{C}}$ and $u_{\mathcal{C}}^a$. Hence, (4), together with equations for material fields, leads to a complete set of equations.

In what follows, we assume (49) to be the case at small distances to the source of gravity and solve the vacuum static spherically symmetric problem with the additional condition (49). On large scales, this conditions is violated [see Eq. (44)].

A general static spherically symmetric metric on the brane has the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (50)$$

It leads to the following nonzero components of the Einstein tensor:

$$-G^t_t = \frac{h'}{rh^2} + \frac{1}{r^2} \left(1 - \frac{1}{h}\right), \quad (51)$$

$$G^r_r = \frac{f'}{rfh} - \frac{1}{r^2} \left(1 - \frac{1}{h}\right), \quad (52)$$

$$G^\theta_\theta = G^\varphi_\varphi = \frac{f''}{2fh} - \frac{f'h'}{4fh^2} - \frac{f'^2}{4f^2h} + \frac{f'}{2rfh} - \frac{h'}{2rh^2}. \quad (53)$$

The interior of static objects (such as stars) is naturally described by the stress–energy tensor of perfect fluid $T_{ab} = \rho u_a u_b + p(h_{ab} + u_a u_b)$. To be compatible with the space-time symmetry, the fluid four-velocity u^a must be aligned with the static Killing vector field ξ^a :

$$u^a = -\sqrt{f} \left(\frac{\partial}{\partial t} \right)^a, \quad (54)$$

so that matter is described by the two functions $\rho(r)$ and $p(r)$, and the coordinate components of its stress–energy tensor are given by

$$T^\alpha_\beta = \begin{pmatrix} -\rho(r), & \vec{0} \\ \vec{0}, & p(r)\delta^i_j \end{pmatrix}. \quad (55)$$

The spherical symmetry and static property imply that the Weyl fluid four-velocity is also aligned with the static Killing vector field:⁵

$$u^a_c = -\sqrt{f} \left(\frac{\partial}{\partial t} \right)^a, \quad (56)$$

so that the Weyl fluid is described by the single function $\rho_c(r)$:

$$m^2 C^\alpha_\beta = \begin{pmatrix} -\rho_c(r), & \vec{0} \\ \vec{0}, & \frac{1}{3}\rho_c(r)\delta^i_j \end{pmatrix}. \quad (57)$$

One should note that $C^r_r = C^\theta_\theta$ as a result of our boundary condition (49) which requires vanishing of the anisotropic stress. The tensor C_{ab} manifests itself directly as dark radiation in the Randall–Sundrum model [see Eq. (9)]; however, in a braneworld model with induced gravity, which is described by the effective equation (4), its influence on the gravitational dynamics is more complicated, and new effects can be expected.

To proceed further, we rewrite Eq. (4) as

$$Q_{ab} = \frac{M^6(1+\beta)}{m^2} E_{ab} + M^6 \left(\frac{1}{m^2} T_{ab} + C_{ab} + \Lambda_{\text{RS}} h_{ab} \right). \quad (58)$$

⁵ The four-velocity of the Weyl fluid certainly will play an important role in a non-static situation.

Here, Q_{ab} is the quadratic expression with respect to the nonzero diagonal components of E_{ab} given by (7). Introducing

$$x = -E^t_t, \quad y = E^r_r, \quad z = E^\theta_\theta \quad (59)$$

and

$$\gamma = \frac{3M^6(1+\beta)}{m^2}, \quad \gamma_1 = 3M^6 \left(\frac{\rho + \rho c}{m^2} - \Lambda_{\text{RS}} \right), \quad \gamma_2 = 3M^6 \left(\frac{p + \frac{1}{3}\rho c}{m^2} + \Lambda_{\text{RS}} \right), \quad (60)$$

we obtain from (58):

$$x^2 - y^2 - z^2 + 2yz = \gamma x + \gamma_1, \quad (61)$$

$$x^2 - y^2 + z^2 + 2xz = \gamma y + \gamma_2, \quad (62)$$

$$x^2 + y^2 + xy + xz - yz = \gamma z + \gamma_2, \quad (63)$$

which is a system of algebraic equations with the components of E^α_β playing the role of unknown variables.

Because of the condition $C^r_r = C^\theta_\theta$, the same constant γ_2 appears on the right-hand side of (62) and (63). This allows us to solve this system of algebraic equations in the most general case. Subtracting (62) from (63), we obtain

$$(y - z)(x + 2y + z + \gamma) = 0, \quad (64)$$

which admits two alternative solutions: $y = z$ or $x + 2y + z + \gamma = 0$.

First, consider the case $y = z$. The solution of (61), (62) is then:

$$-G^t_t - \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}\rho \pm \frac{2}{\ell} \sqrt{3 \left[\frac{\rho + \rho c}{m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}, \quad (65)$$

$$G^r_r + \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}p \mp \frac{6}{\ell} \cdot \frac{\left[\frac{\rho - p + 2\rho c/3}{2m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}{\sqrt{3 \left[\frac{\rho + \rho c}{m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}}, \quad (66)$$

$$G^\theta_\theta + \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}p \mp \frac{6}{\ell} \cdot \frac{\left[\frac{\rho - p + 2\rho c/3}{2m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}{\sqrt{3 \left[\frac{\rho + \rho c}{m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}}. \quad (67)$$

The second solution of (64), namely, $x + 2y + z + \gamma = 0$, leads to the result

$$-G^t_t - \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}\rho \mp \frac{2}{\ell} \cdot \frac{\left[\frac{\rho + 3p + 2\rho c}{2m^2} - \frac{3(1+\beta)^2}{\ell^2} + \Lambda_{\text{RS}} \right]}{\sqrt{\frac{\rho + \rho c}{m^2} - \frac{3(1+\beta)^2}{\ell^2} + \Lambda_{\text{RS}}}}, \quad (68)$$

$$G^r_r + \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}p \pm \frac{2}{\ell} \sqrt{\frac{p+\rho_C/3}{m^2} - \frac{3(1+\beta)^2}{\ell^2} + \Lambda_{\text{RS}}}, \quad (69)$$

$$G^\theta_\theta + \frac{6(1+\beta)}{\ell^2} = \frac{1}{m^2}p \pm \frac{2}{\ell} \cdot \frac{\left[\frac{\rho-p+2\rho_C/3}{2m^2} + \frac{3(1+\beta)^2}{\ell^2} - \Lambda_{\text{RS}} \right]}{\sqrt{\frac{p+\rho_C/3}{m^2} - \frac{3(1+\beta)^2}{\ell^2} + \Lambda_{\text{RS}}}}. \quad (70)$$

It is remarkable that the effective braneworld equations in a static spherically symmetric situation in our model have the form of Einstein equations with a special modification on the right-hand side. We should also note that the system of equations (65)–(67) matches with the two branches of cosmological equations (17) of the braneworld model derived in [29]. This means that writing the components G^α_β for the Friedmann–Robertson–Walker metric and substituting them to the left-hand side of (65)–(67), one obtains the cosmological equations (17). As for the branches described by equations (68)–(70), their connection with cosmology is not clear. Because of this, we restrict our investigation only to the analysis of system (65)–(67). Following the cosmological nomenclature, we call the branch with the lower (‘−’) sign Brane1, and the branch with the upper (‘+’) sign Brane2. These two branches have a geometric origin: they correspond to the two possible branches of solutions in the five-dimensional bulk space [29].

It is convenient to introduce new functions

$$\varrho = \rho \pm 2\xi\rho_L \left(\sqrt{1 + \frac{\rho+\rho_C}{\rho_L}} - 1 \right), \quad (71)$$

$$\mathcal{P} = p \mp 2\xi\rho_L \left(\frac{1 + \frac{\rho-p+2\rho_C/3}{2\rho_L}}{\sqrt{1 + \frac{\rho+\rho_C}{\rho_L}}} - 1 \right), \quad (72)$$

and a new constant

$$\lambda = \frac{6}{\ell^2} \left(1 + \beta \pm \frac{1}{\xi} \right), \quad (73)$$

where ρ_L and ξ are constant parameters defined by

$$\rho_L = \frac{3m^2}{L^2}, \quad \xi = \frac{L}{\ell}, \quad (74)$$

and L is a new length scale

$$\frac{1}{L^2} = \frac{1}{\ell^2} + \frac{\sigma}{3m^2} - \frac{\Lambda}{6}. \quad (75)$$

We assume the expression on the right-hand side of (75) to be positive to avoid singularity in the square root of (71), (72), hence, also in system (65)–(67), for positive values of ρ and

ρ_C .⁶ To avoid a similar singularity in (68)–(70), one would require the opposite condition, namely $1/\ell^2 + \sigma/3m^2 - \Lambda/6 < 0$. Thus, one can think that the choice between the two systems of equations (65)–(67) and (68)–(70) is determined by the value of the expression on the right-hand side of (75). If this expression is equal to zero, then these two systems coincide.

The signs of σ (or β) and Λ are not yet specified, but one should note that the above condition leads to the constraint $\beta > -1/2 + \Lambda\ell^2/12$. Transition to the DGP model in our equations is realized by setting $\sigma = 0$ and $\Lambda = 0$, which implies $\beta = 0$ and $\xi = 1$. In the cosmic-mimicry model with $\beta \approx -\alpha \approx -5/4$, which we discussed in Sec. IV, the value of ξ can be found to be

$$\xi \approx \frac{1}{|1 + \beta|} \approx 4. \quad (76)$$

In principle, the density of the Weyl fluid ρ_C , unlike that of realistic matter density, is not restricted in its sign. However, the above condition of absence of singularity restricts the range of boundary conditions for $\rho_C(r)$ which we shall specify in the next section.

In the new notation (71)–(73), system (65)–(67) is written as follows:

$$-G^t_t = \lambda + \frac{1}{m^2}\varrho, \quad (77)$$

$$G^r_r = -\lambda + \frac{1}{m^2}\mathcal{P}, \quad (78)$$

$$G^\theta_\theta = -\lambda + \frac{1}{m^2}\mathcal{P}. \quad (79)$$

The quantities ϱ and \mathcal{P} play the role of effective energy density and pressure in the usual Einstein equations describing the static spherically symmetric situation, and λ is the effective cosmological constant. The formal solution of (77)–(79) is well known:

$$ds^2 = -e^{2\phi(r)}dt^2 + \frac{dr^2}{1 - 2\mu(r)/r} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (80)$$

where

$$\mu(r) = \frac{\lambda r^3}{6} + \frac{1}{2m^2} \int_0^r \varrho(r') r'^2 dr', \quad (81)$$

$$\phi'(r) = \frac{\mu(r) + r^3\mathcal{P}(r)/2m^2 - \lambda r^3/2}{r[r - 2\mu(r)]}. \quad (82)$$

⁶ These singularities directly correspond to the ‘quiescent’ singularities in braneworld cosmology discussed in [30].

The distributions of the densities of usual matter and Weyl fluid are determined by the energy–momentum conservation and by the analogue of the Tolman–Oppenheimer–Volkoff equation [which, in our case, is a simple consequence of the Bianchi identity applied to (77)–(79)]. In Einstein’s theory, the energy–momentum conservation equation and the Bianchi identity give the same result, which is precisely the Tolman–Oppenheimer–Volkoff equation. In our model, due to the presence of *effective* ϱ and \mathcal{P} on the right-hand side of (77)–(79), which do not coincide with the usual ρ and p , we have two different equations:

$$p' + \phi'(\rho + p) = 0, \quad (83)$$

$$\left(\frac{1}{3} + \zeta\right) \rho'_c + (1 + \zeta)\rho' = \frac{8\rho_c}{3}\phi', \quad (84)$$

where

$$\zeta = \frac{p + \rho_c/3 - \rho_L}{\rho + \rho_c + \rho_L}. \quad (85)$$

For a given equation of state $p = p(\rho)$ and some boundary conditions for $\rho(r)$ and $\rho_c(r)$, one can integrate (at least, numerically) (81), (82), (83), and (84) to obtain the metric and matter density distribution inside a massive object. As for the boundary conditions, it looks natural to specify the values of ρ and ρ_c at the center of the spherically symmetric gravitating body. At the surface of this object, the interior solution matches with the exterior vacuum solution of the above equations.

It should be noted here that Eq. (84) gives the condition of incompressible fluid $\rho'(r) = 0$ for the ordinary matter if we set $\rho_c(r) = 0$, which corresponds to a complete neglect of the tensor C_{ab} in the initial field equations [13, 31]. Allowing for nonzero density of the Weyl fluid replaces this restrictive condition on $\rho(r)$ by a differential equation determining the behavior of $\rho_c(r)$.

We see from (77)–(79) that the effective gravitational constant is given by $8\pi G_N = 1/m^2$. For $\ell \approx 2$ Mpc and $\xi \approx 4$, the value ρ_L in (74) can be estimated to be $\rho_L \approx 10^{-22} \text{ g} \cdot \text{cm}^{-3}$. This density scale lies far below the density of compact objects, such as stars and planets. Thus, inside such objects, we expect only slight modification of Newtonian dynamics due to the bulk effects [$\varrho(r) \approx \rho(r)$ and $\mathcal{P}(r) \approx p(r)$ if $\rho_c(r) \lesssim 1$]. The vacuum situation corresponding to this case will be studied in the next section.

The local disk density of a typical spiral galaxy is $\rho_{\text{disk}} \sim 3\text{--}12 \times 10^{-24} \text{ g} \cdot \text{cm}^{-3}$, and, therefore, the bulk effects inside the disk are significant. The solution of the interior problem

for such objects is of special interest, but is beyond the target of this article. The solution to this problem will hopefully give a relation between the asymptotic rotational velocity v_c and the gravitational mass of a spiral galaxy.

VII. VACUUM SOLUTIONS

A. General equations and boundary condition

In the vacuum, which is defined by the conditions $\rho(r) = 0$ and $p(r) = 0$, the effective matter density $\varrho(r)$ and pressure $\mathcal{P}(r)$ are not necessarily equal to zero due to the possible presence of the Weyl fluid component $\rho_C(r)$:

$$\varrho_{\text{vac}}(r) = \pm 2\xi\rho_L \left(\sqrt{1 + \rho_C(r)/\rho_L} - 1 \right), \quad (86)$$

$$\mathcal{P}_{\text{vac}}(r) = \mp 2\xi\rho_L \left(\frac{1 + \rho_C(r)/3\rho_L}{\sqrt{1 + \rho_C(r)/\rho_L}} - 1 \right). \quad (87)$$

One should note that the effective equation of state in this case

$$\frac{\mathcal{P}_{\text{vac}}}{\varrho_{\text{vac}}} = \frac{1}{3} \left(\frac{2}{\sqrt{1 + \rho_C/\rho_L}} - 1 \right) = \frac{1}{3} \cdot \frac{1 \mp \varrho_{\text{vac}}/(2\xi\rho_L)}{1 \pm \varrho_{\text{vac}}/(2\xi\rho_L)} \quad (88)$$

differs from that of radiation. Specifically, $\mathcal{P}_{\text{vac}}/\varrho_{\text{vac}} < 1/3$ if $\rho_C(r)$ is positive.

In addition to equations (77), (78), we can use Eq. (84) instead of (79) in the vacuum case:

$$\frac{\rho_C - \rho_L}{\rho_C + \rho_L} \rho'_C = 4\rho_C \phi'. \quad (89)$$

The key issue concerns the choice of the boundary condition for $\rho_C(r)$. As we noted above, it is natural to specify the value of ρ_C at the center of the spherically symmetric gravitating body. However, as the interior problem is rather complicated and because we are not going to consider it in this paper, it is reasonable to parameterize the boundary condition by the value of ρ_C in the vacuum at some radius R . For a massive body, this could be the radius of its surface, from which the vacuum solution starts. Thus $\rho_R \equiv \rho_C(R)$ is a new free parameter of our model, defining the boundary condition for the Eq. (89) in vacuum. It is restricted by the condition $\rho_R > -\rho_L$, which is required for the expression under the square root of the right-hand side of (86) and (87) to be positive.

Eq. (89) can be solved exactly, relating the energy density of the Weyl fluid $\rho_C(r)$ and the function $\phi(r)$ entering metric (80):

$$\frac{\rho_C(r)}{[\rho_L + \rho_C(r)]^2} = \frac{\rho_R}{(\rho_L + \rho_R)^2} e^{4[\phi_R - \phi(r)]}, \quad (90)$$

where $\phi_R \equiv \phi(R)$. If the boundary value $\rho_R = 0$, then $\rho_C(r) \equiv 0$ for $r \geq R$. In this case, $\varrho_{\text{vac}}(r) = \mathcal{P}_{\text{vac}}(r) = 0$ [see (86) and (87)], and we obtain the usual vacuum Einstein equations with a cosmological constant λ . Thus, our braneworld model admits a Schwarzschild-(A)dS₄ space as an exact vacuum solution. The interior counterpart of this exterior solution in the context of the Randall–Sundrum braneworld model was analyzed in [14, 17].

To solve equations (77) and (78) in the vacuum, we introduce the function

$$\Delta(r) = \frac{e^{2\phi(r)}}{1 - 2\mu(r)/r}. \quad (91)$$

Then one has

$$\frac{2\mu'}{r^2} = \lambda \pm \frac{2\xi\rho_L}{m^2} \left(\sqrt{1 + \frac{\rho_C}{\rho_L}} - 1 \right), \quad (92)$$

$$\frac{1}{r} \left(1 - \frac{2\mu}{r} \right) \frac{\Delta'}{\Delta} = \pm \frac{4\xi}{3m^2} \frac{\rho_C}{\sqrt{1 + \rho_C/\rho_L}}. \quad (93)$$

This nonlinear system in its full generality looks quite complicated. But, in fact, one can note that the gravitational field is usually weak in the neighborhood of an astrophysical object. So, in analysis of (92), (93), we restrict ourselves only by the Newtonian approximation.

B. Newtonian approximation

First, we note that the function $\phi(r)$ is an analogue of the Newtonian gravitational potential in this case. Under the condition

$$|\phi(r)| \ll 1, \quad (94)$$

Eq. (90) implies an almost constant value of $\rho_C(r)$:

$$\rho_C(r) \approx \rho_R. \quad (95)$$

In this approximation, Eq. (92) can be integrated:

$$1 - \frac{2\mu(r)}{r} = 1 - \frac{r_g}{r} - \frac{\lambda r^2}{3} \mp \frac{2\xi r^2}{L^2} \left(\sqrt{1 + \frac{\rho_R}{\rho_L}} - 1 \right), \quad (96)$$

where r_g is the integration constant and is an analogue of the Schwarzschild radius in general relativity.

The Newtonian approximation requires also the condition $|\mu(r)/r| \ll 1$. Assuming it to be satisfied, we can solve Eq. (93):

$$\Delta(r) \approx e^{\pm r^2/r_N^2} \approx 1 \pm \frac{r^2}{r_N^2}, \quad (97)$$

where

$$r_N^2 = \frac{L^2}{2\xi} \cdot \frac{\sqrt{1 + \rho_R/\rho_L}}{|\rho_R/\rho_L|} \quad (98)$$

determines the radius at which the Newtonian approximation fails. The integration constant that arises in solving (93) is absorbed by rescaling the time coordinate. The parameter r_N is unambiguously defined by ρ_R and can be taken as a new free parameter of our model. The Newtonian potential in this case is defined by

$$e^{2\phi(r)} \approx 1 + 2\phi(r) \approx 1 - \frac{r_g}{r} - \frac{\lambda r^2}{3} \pm \frac{2\xi r^2}{L^2} \cdot \frac{(\sqrt{1 + \rho_R/\rho_L} - 1)}{\sqrt{1 + \rho_R/\rho_L}}, \quad (99)$$

and the region of applicability of the above result is

$$r_g \ll r \ll r_N. \quad (100)$$

One can easily verify that, under condition (100), our initial assumptions $|\phi(r)| \ll 1$ and $|\mu(r)/r| \ll 1$ are satisfied. If ρ_R/ρ_L is of order unity, then $r_N \sim L/\sqrt{\xi}$. Equations (96) and (99) then describe the Schwarzschild metric in the region $r^3 \lesssim \xi r_g \ell^2$. For ξ given by (76), this condition is compatible with condition (16), thus justifying our initial assumption that bulk effects is insignificant at sufficiently small scales.

A remarkable feature of our model in comparison with the Randall–Sundrum braneworld model is the appearance of a new length scale ℓ [which goes to zero in the limit $m \rightarrow 0$; see (75)] defining the radius $r_* \sim r_g \ell^2$ [see (16)] up to which the Newtonian gravity works. Beyond the radius r_* , our model predicts transition to gravity with different properties, for which the influence of the projection of the bulk Weyl tensor C_{ab} on the dynamics can be significant, thus allowing to explain the flatness of galactic rotation curves, as we have demonstrated in Sec. V.

VIII. CONCLUSION

In this paper, we continued the analysis of a generic braneworld model with induced gravity described by action (1). It was shown previously in [10] that a low-density braneworld can mimic the expansion properties of the LCDM model. In particular, a universe consisting solely of baryons with $\Omega_b \simeq 0.04$ can mimic the LCDM cosmology with a much larger ‘effective’ value of the matter density $\Omega_m^{\text{LCDM}} \simeq 0.2\text{--}0.3$. This property fixes the parameter β defined in (5) to be $\beta \approx -5/4$.

The general analysis of the braneworld model with induced gravity performed in Sec. III demonstrates that, in the neighborhood of the source, at distances smaller than r_* given by (16), the solution is determined mainly by the quadratic part Q_{ab} in (4), which means that it respects the ‘bare’ Einstein equation $m^2 G_{ab} - T_{ab} = 0$ to a high precision. This effect is sometimes described as the ‘gravity filter’ of the DGP model [3], which screens the scalar graviton in the neighborhood of the source making the gravity effectively Einsteinian.

At large distances, the modified gravitational field equations can be used to explain the properties of the galactic rotation curves. Using the smallness of the rotational velocity v_c , we have developed a perturbative scheme for reconstructing the metric in this region. In the leading order of expansion, in the region given by (41), our result reproduces that obtained for the Randall–Sundrum braneworld model in [6, 7]. The vacuum gravity in this case can be approximately described by the Einstein equations with the traceless projection C_{ab} of the bulk Weyl tensor to the brane playing the role of some effective stress–energy tensor of galactic fluid (44). The left inequality in (41) is satisfied in a real spiral galaxy such as our Milky Way with the distance $r \sim 3$ kpc at which the rotation velocity becomes constant, $v_c \sim 7 \times 10^{-4}$, if $\ell \lesssim \sqrt{|1 + \beta|} v_c^{-1} r \simeq 2$ Mpc. In this paper, we restricted ourselves to this case only. In the opposite case, one has to solve a nonlinear differential equation (29).

The role of dark matter in the braneworld model is played by the traceless tensor of the Weyl fluid. Using the methods of [12], in which it was proposed to use the existing data on rotation curves and lensing measurements to constrain the equation of state of galactic fluid, we have shown that deflection angles in our model are about 75% of the usual predictions based on dark matter. This result confirms the analysis done in [5] and allows for a direct verification of the braneworld paradigm.

In the region near to the gravitational object, we derive a closed system of equations for

a static spherically symmetric situation in the case of zero anisotropic stress in the tensor C_{ab} . We demonstrate that the effective braneworld equations in this case have the form of Einstein equations with a special modification on the right-hand side [see (77)–(79)]. We find the Schwarzschild metric to be an approximate vacuum solution of these equations at distances well below the scale r_* , in agreement with the general consideration of Sec. III.

Expression (16) for r_* generalizes the length scale [11] of the DGP model, below which nonlinear effects become important, to the case of nonzero brane tension (nonzero β) and bulk cosmological constant satisfying the Randall–Sundrum constraint $\Lambda_{\text{RS}} = 0$. For the parameter values $\beta \approx -5/4$ and $\ell \lesssim 2$ Mpc determined from the explanation of dark-matter phenomena, the value of r_* for the Sun turns out to be of the order of 1 pc, which is quite large so that the solar-system experiments are in accord with the general-relativistic expectations.

The density scale (74) which appears in our model is estimated to be $\rho_L \approx 10^{-22} \text{ g} \cdot \text{cm}^{-3}$. This value lies far below the density of compact objects, such as stars and planets. Thus, inside such objects, we expect only slight modification of Newtonian dynamics due to the bulk effects. However, the local disk density of a typical spiral galaxy is $\rho_{\text{disk}} \sim 3\text{--}12 \times 10^{-24} \text{ g} \cdot \text{cm}^{-3}$, and, therefore, the bulk effects inside the disk should be significant. The interior problem for such objects remains a challenge for future investigation. The solution to this problem will hopefully give a relation between the asymptotic rotational velocity v_c and the mass of a spiral galaxy. It may help to verify the Tully–Fisher relation and extend our results to a wider class of astrophysical objects such as ellipticals and ultra compact dwarf galaxies.

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